we call type (1,1). Similarly we define the rearrangements of type (2,2): $2T^2 \to Q \to 2T^2$; (1,2): $T^2 \to U \to 2T^2$; (0,1): $\varnothing \to S \to T^2$. The last two rearrangements occurring in the reverse order we denote, respectively, by the symbols (2,1) and (1,0). The notation (1:1) denotes a continuous deformation of the connected component of the integral manifold on which there are no critical points. The symbols of simultaneously occurring rearrangements are connected by a plus sign, or will indicate an integral multiplier, if they are identical.

Let us now enumerate the bifurcation sequence taking place along the dash-dot arrows in Fig.1: a) $2(0,1), (2,1), (1,2), (2,2), 2(1,1)$; b) $(0,1), (1:1)+(0,1)$; c) $(0,1), (1,2)$. The transition from component 2 to component 5 from above (Fig.3) is accompanied by bifurcation $2(1:1)+2(0,1)$, and from below by $2(1,2)$. In passing from component 5 to component 3 we have bifurcation $2(2,1)$, and when emerging from component 5 into region $k<0$ we obtain bifurcation $4(1,0)$.

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ANALYTIC SOLUTIONS IN THE THEORY OF COAGULATING SYSTEMS WITH SINKS*

A.A. LUSHNIKOV and V.N. PISKUNOV

Analytic solutions are derived for the problem of the evolution of the mass spectrum of three models of coagulating systems with three-dimensional uniform sinks. The case when the rate of drainage of particles with masses greater than some critical value G is higher compared with the rate of an individual act of coalescence is considered, and the problem is reduced to the consideration of a coagulation process without sinks, but where coagulation of particles of mass greater than G is forbidden. Coagulation kernels that are a) independent of the mass of the colliding particles, b) proportional to the sum, and c) equal to the product of masses of colliding particles are considered. Exact expressions are obtained for the dependence of the coagulating particles mass spectrum and for the sediment, and their asymptotic form in the limit when G is large is analyzed.

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1. A three-dimensional uniformly coagulating dispersed system is defined by the mass spectrum $c_g(t)$, i.e. the concentration of the dispersed phase particles of mass g at the instant of time t. It is often convenient to assume that each particle consists of unit mass monomers. As the latter, we may select individual molecules that constitute the particle. Then g is the number of molecules in a particle whose mass is measured on molecular mass units. This definition is used below.

The reason for the time evolution of the mass spectrum in coagulating systems are the separate acts of particle coalescence, whose velocity $K(l_1, l_2)$ (the coagulation kernel) is assumed to be a known function of the masses l_1 and l_2 of the colliding particles, determined from the independently solved problem of the relative motion of two particles in the carrier medium.

Knowing $K(l_1, l_2)$ we can set up an equation (the Smoluchowski equation) controlling the time evolution of the mass spectrum. Details of the derivation can be found in /1, 2/, the essence of which is that the rate of spectrum variation $d_t c_g$ is equated to the difference of the rate of all coalescence processes resulting in the appearance of g-mers less the rate of their loss produced by the sticking of g-mers to all other particles. The equation has the form

$$\frac{dc_g}{dt} = \frac{1}{2}\sum_{l=1}^{g-1} K(g-l, l)\, c_{g-l} c_l - c_g \sum_{l=1}^{\infty} K(g, l)\, c_l \tag{1.1}$$

The following analytic solutions exist for three models with kernels (see /1–3/ and references there):

$$K_1(l_1, l_2) = 2, \quad K_2(l_1, l_2) = 2(l_1 + l_2 - 1), \quad K_3(l_1, l_2) = l_1 l_2 \tag{1.2}$$

Exact solutions are obtained below for three more models with coagulation kernels of the form:

$$K(l_1, l_2) = K_i(l_1, l_2)\, \Theta_G(l_1)\, \Theta_G(l_2) \tag{1.3}$$

Here $K_i(l_1, l_2)$ is any of the kernels (1.2), and $\Theta_G(l)$ is the step function: $\Theta_G(l) = 1$ when $l \leqslant G$ and $\Theta_G(l) = 0$ when $l > G$. The kernels (1.3) correspond to the fact that the particles of mass greater than G cannot coagulate. The physical meaning of the model considered here is that particles of mass greater than G are momentarily taken out of the system and fall into the sediment whose spectrum is $c_g(t)\, \bar{\Theta}_G(g)$ $(\bar{\Theta}_G(g) \equiv 1 - \Theta_G(g))$. Thus the generating factor simulates the particle sink.

2. In coagulating systems with kernels of the form (1.3) the mass spectrum develops in a fairly specific manner. In the range of masses $1 \leqslant g \leqslant G$ the coagulation proceed regularly, i.e. the concentration of g-mers decreases with time to zero. In the range $G < g \leqslant 2G$ particles incapable of further evolution collect in the sediment. We will denote the respective spectra by $c_g^-(t)$ and $c_g^+(t)$. Then

$$c_g^-(t) \equiv c_g(t)\, \Theta_G(g), \quad c_g^+(t) \equiv c_g(t)\, \bar{\Theta}_G(g) \tag{2.1}$$

Multiplying (1.1), respectively, by $\Theta_G(g)$ and $\bar{\Theta}_G(g)$, we obtain the following equations which control the time evolution of the active part of the spectrum $c_g^-(t)$ and the spectrum of the sediment $c_g^+(t)$

$$\frac{dc_g^-(t)}{dt} = \frac{1}{2}\Theta_G(g) \sum_{l=1}^{g-1} K_i(g-l, l)\, c_{g-l}^- c_l^- - c_g^- \sum_{l=1}^{G} K_i(g, l)\, c_l^- \tag{2.2}$$

$$\frac{dc_g^+}{dt} = \frac{1}{2}\bar{\Theta}_G(g) \sum_{l=1}^{g-1} K_i(g-l, l)\, c_{g-l}^- c_l^- \tag{2.3}$$

Thus to determine the spectrum it is necessary to solve (2.2), after which the sediment spectrum is determined by integrating (2.3).

Below, a complete solution is obtained of the problem of coagulation in model systems of initially monodispersed particles, i.e. the initial conditions for (2.2) and (2.3) are selected in the form ($\delta_{g,l}$ is the Kronecker delta)

$$c_g(0) = \delta_{g,1} \tag{2.4}$$

3. First, the case of coagulation kernel $K = 2\Theta_G(l_1)\, \Theta_G(l_2)$ is considered. For the initial condition (2.4) it is convenient to use the variables ν, τ introduced by the equations /2/

$$c_g^{\pm}(\tau) = \nu^{\pm}(\tau)\, c_1(\tau) \tag{3.1}$$

$$\tau = \int_0^t c_1(t')\,dt'$$

Subsituting (3.1) into (2.2) and (2.3) we obtain

$$\frac{d\nu_g^-}{d\tau} = \Theta_G(g)\sum_{l=1}^{g-1}\nu_{g-l}^-\nu_l^- \tag{3.2}$$

$$\frac{dc_g^+}{d\tau} = \overline{\Theta}_G(g)\,c_1\sum_{l=1}^{g-1}\nu_{g-l}^-\nu_l^-$$

The solution of the first equation of (3.2) with initial condition (2.4) has the form

$$\nu_g^-(\tau) = \tau^{g-1} \tag{3.3}$$

Substituting (3.3) into the second equation of (3.2) and integrating we obtain

$$c_g^+(\tau) = (2G - g + 1)\int_0^\tau \xi^{g-2} c_1(\xi)\,d\xi \tag{3.4}$$

When deriving (3.4) the following obvious relation was used:

$$l_G(g)\sum_{l=1}^{g-1}\Theta_G(g-l)\,\Theta_G(l) = 2G - g + 1$$

The dependence on τ of the monomer concentration c_1 is determined by an equation resulting from (2.2) after substituting (3.1)

$$\frac{dc_1}{d\tau} = -2c_1(\tau)\sum_{l=1}^{G}\nu_l^-$$

Integration of this equation using (3.3) yields

$$c_1(\tau) = \exp[-2I(\tau)], \quad I(\tau) \equiv \int_0^\tau \frac{1-\eta^G}{1-\eta}\,d\eta \tag{3.5}$$

Equation (3.5) together with (3.1), (3.3) and (3.4) enables us finally to determine $c_g^-(\tau)$ and $c_g^+(\tau)$.

The change from the variable τ to real time t is made using the following formula, which follows directly from (3.2):

$$t(\tau) = \int_0^\tau \frac{d\xi}{c_1(\xi)} = \int_0^\tau \exp 2I(\xi)\,d\xi \tag{3.6}$$

Note that unlike the case of $G = \infty$, $\tau = \infty$ corresponds to the instant $t = \infty$ and not $\tau = 1$. This follows from the fact that the integrand in (3.6) has no singular points when ξ is finite. This information enables us to determine the final sediment spectrum

$$c_g^+(t = \infty) = c_g^+(\tau = \infty) = (2G - g + 1)\int_0^\infty \xi^{g-2} c_1(\xi)\,d\xi \tag{3.7}$$

4. Let us analyze the asymptotic form of the results obtained when $G \gg 1$. We begin with the case when $\tau < 1$, $1 - \tau \gg G^{-1}$. Neglecting τ^G, compared with 1, we find from (3.5), (3.6) and (3.7)

$$\tau = \frac{t}{t+1}, \quad c_1(t) = \frac{1}{(t+1)^2}, \tag{4.1}$$

$$c_g^-(t) = \frac{t^{g-1}}{(t+1)^{g+1}}, \quad c_g^+(t) \approx \frac{2G-g}{g}\,\frac{t^{g-2}}{(t+1)^g}$$

The first three formulas of (4.1) do not differ from those when $G = \infty$. This has a simple explanation: the spectrum has not developed sufficiently to feel the constraints on the coagulation coefficient. Almost the whole mass of the dispersed phase is concentrated in the active part of the spectrum. The sink mass is small, like τ^G. The basic events related to the transition of the active fraction to the sink develop in the neighbourhood of the point $\tau = 1$. This clearly shows in the asymptotic evaluation of the integral that defines the final sediment spectrum.

To evaluate the integral on the right-hand side of (3.7), we divide the integration region into three segments $[0, 1-\Delta]$, $[1-\Delta, 1+\Delta]$, $[1+\Delta, \infty)$ selecting the quantity Δ so that $G\Delta \to \infty$ as $G \to \infty$, but $G\Delta^2 \to 0$. The last condition allows the approximation $(1+\Delta)^G \approx e^{G\Delta}$ to be used. It can be shown that the integrals over the first and second segments are small, like $e^{-G\Delta}$, so that it remains merely to evaluate the integral

$$\int_{1-\Delta}^{1+\Delta} \xi^{g-2} \exp[-2I(\xi)]\, d\xi = \exp[-2I(1)] \int_{1-\Delta}^{1+\Delta} \xi^{g-2} \exp[-2(I(\xi) - I(1))]\, d\xi \tag{4.2}$$

When G is large ($\mathbf{C}$ is Euler's constant)

$$I(1) = \sum_{g=1}^{G} \frac{1}{g} \approx \ln G + \mathbf{C}$$

To calculate the difference $I(\xi) - I(1)$ we make the change $\xi = 1 + x\Delta$. Then

$$I(\xi) - I(1) \approx \int_0^{Gx\Delta} \frac{e^{\xi} - 1}{\xi}\, d\xi \equiv Q(Gx\Delta) \tag{4.3}$$

After the change, and substituting (4.3) into (4.2) we obtain

$$\int_{1-\Delta}^{1+\Delta} \xi^{g-2} \exp[-2I(\xi)]\, d\xi \approx \Delta \int_{-1}^{1} \exp[gx\Delta - 2Q(gx\Delta)]\, dx =$$

$$\frac{1}{G} \int_{-G\Delta}^{G\Delta} \exp[(1+\lambda)s - 2Q(s)]\, ds \approx \frac{1}{G} \int_{-\infty}^{\infty} \exp[(1+\lambda)s - 2Q(s)]\, ds$$

where the parameter λ is introduced for convenience by the formula

$$g = G(1+\lambda), \quad 0 \leqslant \lambda \leqslant 1 \tag{4.4}$$

Finally, the asymptotic form of the final spectrum of the sediment is

$$c_g^+(\infty) = \frac{1}{G^2}(1-\lambda) e^{-2\mathbf{C}} \int_{-\infty}^{\infty} \exp\left[(1+\lambda)s - 2\int_0^s \frac{e^{\xi} - 1}{\xi}\, d\xi\right] ds \tag{4.5}$$

The denumerable concentration of particles in the sediment is obtained by integrating the spectrum (4.5)

$$N^+(\infty) = \int_G^{2G} c_g^+\, dg = \frac{e^{-2\mathbf{C}}}{G} \int_{-\infty}^{\infty} \frac{e^s}{s^2}(e^s - 1 - s) e^{-2Q(s)}\, ds \approx \frac{0{,}7923}{G} \tag{4.6}$$

The condition for normalization on unit of mass was checked numerically.

The relation $R_G(\lambda) \equiv G^2 c_g^+(\infty)$ is tabulated below together with the results of numerical calculations of $R_G(\lambda)$ using the exact formula (3.7) for $G = 50$ and $G = 100$, which enables us to follow the convergence of the spectrum to the asymptotic limit (4.5):

λ	0	0,1	0,3	0,5	0,7	0,9	1,0
$R_\infty(\lambda)$	2,64	1,91	1,04	0,574	0,287	0,0847	0
$R_{100}(\lambda)$	—	1,92	1,05	0,582	0,294	0,0921	0,008
$R_{50}(\lambda)$	—	1,93	1,06	0,588	0,300	0,0989	0,016

5. We will now consider the model $K = 2(l_1 + l_2 - 1)\, \Theta_G(l_1)\, \Theta_G^-(l_2)$. The equation of the active part of the spectrum in the variables ν, τ has the form

$$\frac{dc_1}{d\tau} = -2c_1 \sum_{l=1}^{G} l\nu_l^- \tag{5.1}$$

$$\frac{d\nu_g^-}{d\tau} = (g-1)\Theta_G(g)\left[\sum_{l=1}^{g-1} \nu_{g-l}^- \nu_l^- - 2\nu_g^- \sum_{l=1}^{G} \nu_l^-\right] \tag{5.2}$$

The equation for the sediment spectrum follows from (2.3)

$$\frac{dc_g^+}{d\tau} = (g-1)\bar{\Theta}_G(g)\, c_1 \sum_{l=1}^{g-1} \nu_{g-l}^- \nu_l^- \tag{5.3}$$

We will seek a solution of (5.2) in the form /2/

$$\nu_g^- = r_g h^{g-1}(\tau) \tag{5.4}$$

Substituting (5.4) into (5.2) and separating the variables, we obtain

$$\frac{dh}{d\tau}+2\sum_{l=1}^{G}r_l h^l=1, \quad h(0)=0 \tag{5.5}$$

$$r_g=\sum_{l=1}^{g-1}r_{g-l}r_l \tag{5.6}$$

To solve (5.6) we introduce the generating function

$$F(z)=\sum_{g=1}^{\infty}r_g z^g$$

Assuming that $r_1 = 1$ (a corollary of initial condition (2.4)), from (5.6) we have

$$F^2-F+z=0$$

Solving this equation, we obtain

$$r_g=2\frac{(2g-3)!}{(g-2)!\,g!} \tag{5.7}$$

Let us now establish the connection between $h(\tau)$ and $c_1(\tau)$. Differentiating (5.5) with respect to τ and using (5.1) we obtain $c_1' = c_1 h''/h'$. Integrating this equation, we obtain

$$c_1 = h'(\tau) \tag{5.8}$$

which enables (5.3) to be integrated, giving

$$c_g^+(\tau)=\overline{\Theta}_G(g)\,h^{g-1}(\tau)\sum_{l=1}^{g-1}r_{g-l}r_l \tag{5.9}$$

The spectrum of the active part is

$$c_g^-(\tau) = r_g h'(\tau)\,h^{g-1} \tag{5.10}$$

Equations (5.5), and (5.7)–(5.10) provide the complete solution of the problem.

6. Let us analyze the results obtained in the limit of large G. For this we shall need the asymptotics

$$r_g \text{ and } \overline{\Theta}_G(g)\sum_{l=1}^{g-1}r_{g-l}r_l$$

Using Stirling's formula, we obtain

$$r_g \approx 4^{g-1}(\pi g^3)^{-1/2} \tag{6.1}$$

Then by definition (4.4) we have

$$\overline{\Theta}_G(g)\sum_{l=1}^{g-1}r_{g-l}r_l\approx\int_{g-G}^{G}r_{g-l}r_l\,dl=\frac{4^{g-1}(1-\lambda)}{\pi G^2(1+\lambda)^2\sqrt{\lambda}} \tag{6.2}$$

The next step is to solve (5.5) for the function $h(\tau)$. From /2/ we know the result when $G=\infty$

$$h(\tau)=\begin{cases}\tau-\tau^2, & \tau\leqslant {}^1/_2\\ {}^1/_4, & \tau>{}^1/_2, \ (\tau={}^1/_2(1-e^{-2t}))\end{cases} \tag{6.3}$$

This result can be used when ${}^1/_2-\tau\lesssim G^{-1}$. The value $\tau={}^1/_2$ corresponds to the completion of coagulation when $G=\infty$. All this means that the spectrum of the active part is exponentially small when $g\sim G$, and the limitation with respect to g is unimportant. The number of particles in the sediment is also exponentially small. Its spectrum may be obtained by combining (6.2), (6.3) and (5.9)

$$c_g^+(\tau)=(\tau-\tau^2)^{g-1}\frac{4^{g-1}}{\pi G^2}\frac{(1-\lambda)}{(1+\lambda)^2\sqrt{\lambda}} \tag{6.4}$$

At the instant $\tau=\tau_0$, when active transition of particles to the sediment begins (the finiteness of G then becomes important), the quantity $h(\tau_0)$ becomes equal to 1/4 and then, as will be shown below, begins to exceed the value mentioned (by a quantity of the order of G^{-1}). When $h(\tau_0)={}^1/_4$ we obtain $h'(\tau_0)$ from (5.5) and by the same token c_1. For this we note that when $\tau=\tau_0$

$$\sum_{l=1}^{G} 4^{-l} r_l = \sum_{l=1}^{\infty} 4^{-l} r_l - \sum_{l=G+1}^{\infty} 4^{-l} r_l \approx \frac{1}{2} - \frac{1}{4\sqrt{\pi}} \int_G^{\infty} g^{-3/2} dg = \frac{1}{2} - \frac{1}{2\sqrt{\pi G}} \tag{6.5}$$

Hence $h' = c_1 = (\pi G)^{-1/2}$. Thus at that moment the spectra take the form

$$c_g^-(\tau_0) = \frac{1}{\pi\sqrt{G}} g^{-3/2}, \quad c_g^+(\tau_0) = \frac{1-\lambda}{\pi G^2 (1+\lambda)^2 \sqrt{\lambda}} \tag{6.6}$$

The mass M^- of the active part of the spectrum, and the mass M^+ of the sediment are

$$M^-(\tau_0) = \int_0^G g c_g^-(\tau_0)\, dg = \frac{2}{\pi} \tag{6.7}$$

$$M^+(\tau_0) = \int_G^{2G} g c_g^+(\tau_0)\, dg = \frac{1}{\pi} \int_0^1 \frac{(1-\lambda)\, d\lambda}{(1+\lambda)\sqrt{\lambda}} = 1 - \frac{2}{\pi}$$

The condition of conservation of mass $M^+ + M^- = 1$ is satisfied.

Let us now find the final sediment spectrum. For this the solution of (5.5) is required when $\tau > 1/2$. It is independent of τ, and we will seek it in the form

$$h(\tau) = \frac{1}{4} \exp \frac{\xi_0}{G} \tag{6.8}$$

The equation for ξ_0 is obtained after substituting (6.8) into (5.5), making the transformations

$$\sum_{l=1}^{G} r_l h^l = \sum_{l=1}^{G} 4^{-l} r_l + \sum_{l=1}^{G} 4^{-l} r_l [\exp(\xi_0 l/G) - 1]$$

and replacing the sums by the integrals (see (6.5))

$$\sqrt{\xi_0} \int_0^{\xi_0} \frac{dx}{x^{3/2}} (e^x - 1) = 2 \tag{6.9}$$

Numerical calculation gives $\xi_0 \approx 0.8540$. Now, combining (5.9), and the second formula (6.6) and (6.8), we obtain

$$c_g^+(\infty) = \frac{1}{\pi G^2} \exp[\xi_0 (1+\lambda)] \frac{1-\lambda}{(1+\lambda)^2 \sqrt{\lambda}} \tag{6.10}$$

Normalization of the total mass to unity was checked numerically. The denumerable sediment concentration is

$$N^+(\infty) = \int_G^{2G} c_g^+(\infty)\, dg = \frac{\xi_0}{G}$$

This completes the analysis of the second model.

7. The solution for the model $K = l_1 l_2 \Theta_G(l_1) \Theta_G(l_2)$ proves to be in many respects similar to the previous ones. For the active part of the spectrum we obtain in the variables ν, τ

$$\frac{dc_1}{d\tau} = -c_1 \sum_{l=1}^{G} l \nu_l^-(\tau) \tag{7.1}$$

$$\frac{d\nu_g^-}{d\tau} = \frac{1}{2} \Theta_G(g) \sum_{l=1}^{g-1} (g-l)\, l \nu_{g-l}^- \nu_l^- - (g-1) \nu_g^- \sum_{l=1}^{G} l \nu_l^- \tag{7.2}$$

For the sediment spectrum we have

$$\frac{dc_g^+}{d\tau} = \frac{1}{2} \Theta_G(g)\, c_1(\tau) \sum_{l=1}^{g-1} (g-l)\, l \nu_{g-l}^- \nu_l^- \tag{7.3}$$

Equation (7.2) is solved by separating the variables, i.e. the solution is sought in the form (5.4). For h' and r_g we obtain other equations

$$h' + \sum_{l=1}^{G} l r_l h^l = 1, \quad (g-1) r_g = \frac{1}{2} \sum_{l=1}^{g-1} (g-l)\, l r_{g-l} r_l \tag{7.4}$$

Multiplying the first of them by z^g and summing with respect to g, we obtain

$$zF' - F = zFF', \quad \text{or} \quad Fe^{-F} = z\left(F(z) \equiv \sum_{g=1}^{\infty} g r_g z^g\right)$$

From this, using contour integration, making the change of variable $z \to F$ and taking into acount the equations for F and F', we obtain

$$r_g = \frac{1}{2\pi i g} \oint \frac{F}{z^{g+1}} dz = \frac{1}{2\pi i g} \oint \frac{F\,dF}{z^{g+1}F'} = \frac{1}{2\pi i g} \oint \frac{(1-F)\,e^{gF}\,dF}{F^g} = \frac{g^{g-2}}{g!} \tag{7.5}$$

The connection between $c_1(\tau)$ and $h(\tau)$ is somewhat more complicated than in the preceeding case. Namely, from (7.1) and the first of (7.4) we obtain $d_\tau \ln(c_1/h) = -1/h$. Changing from τ to t using the second of (3.1), we obtain

$$d_t\,(c_1/h) = -(c_1/h)^2$$

Hence

$$c_1 = \frac{h}{t} \tag{7.6}$$

The spectrum is thus again expressed in terms of the single function h

$$c_g^- = \frac{g^{g-2}}{g!}\,\frac{h^g}{t} \tag{7.7}$$

$$c_g^+(t) = \frac{1}{2}\,\Theta_G(g) \sum_{l=1}^{g-1} (g-l)\,l r_{g-l} r_l \int_0^t \frac{h^g(t')\,dt'}{(t')^2} \tag{7.8}$$

This completes the exact analysis.

8. To investigate the behaviour of the solution for large G we again need the asymptotic form of the coefficients in (7.7) and of the sum in (7.8)

$$r_g \approx \frac{e^g}{\sqrt{2\pi g^5}} \tag{8.1}$$

$$\overline{\Theta}_G(g) \sum_{l=1}^{g-1} (g-l)\,l r_{g-l} r_l \approx \frac{e^g}{\pi G^3}\,\frac{(1-\lambda)}{(1+\lambda)^3\sqrt{\lambda}} \tag{8.2}$$

Up to the instant when then cut-off factor G begins to affect the form of the coagulation spectrum, the function $h(\tau)$ is easily determined. As $G \to \infty$, from (7.4) we obtain $h_\tau' + F(h) = 1$. Differentiating $Fe^{-F} = h$ with respect to h, we obtain $(1-F) = e^F F_h'$ and substituting it into the first equation, we obtain $F_\tau' = e^F$. Then we have

$$h(\tau) = \begin{cases} (1-\tau)\ln\dfrac{1}{1-\tau}, & \tau < 1 - e^{-1} \\ e^{-1}, & \tau > 1 - e^{-1} \end{cases} \tag{8.3}$$

Changing from τ to t using (7.6) and the second of (3.1), we obtain

$$\tau = 1 - e^{-t} \tag{8.4}$$

This relation holds up to the instant $t = 1$. At $t = 1$ we have $h(1) = e^{-1}$, the spectrum of the active fraction becomes exponential, and the effect of the cut-off factor G begins to be substantial. In an infinite system at this instant a superparticle is formed, and the Smoluchowski equation is no longer applicable /4/. The finiteness of G radically alters the situation. At $t > 1$ the particles begin to drop into the sediment. Up to $t = 1$, as seen from (7.7) and (8.1), almost the whole mass is concentrated in the active part of the spectrum. Even at the critical instant $t = 1$, the difference of M^- from unity is of the order of $G^{-1/2}$. Then an appreciable sediment is formed at once.

To trace the above process it is necessary to solve the first of (7.4) with $\tau > 1 - e^{-1}$. This has already been done above. This equation, except for the coefficients, is the same as (5.5), and the structure of its solution is the same

$$h(\tau) = e^{-1}\exp[\xi_0 g/G] \tag{8.5}$$

The constant ξ_0 is, as previously, given by (6.9). The active part of the spectrum when $t > 1$ is

$$c_g^- = \frac{1}{t\sqrt{2\pi g^5}}\exp(\xi_0 g/G) \tag{8.6}$$

To calculate the sediment spectrum it is necessary to evaluate the integral on the right-hand side of (7.8). We divide it into the sum of two integrals: an integral along the segment $[0, 1]$ and an integral along $[1, t]$. For the first we obtain

$$\int_0^1 h^g(t)\, t^{-2}\, dt = \int_0^1 t^{g-2} e^{-gt}\, dt \approx \frac{e^{-g}}{2} \sqrt{\frac{2\pi}{g}} \sim e^{-g} g^{-1/2}$$

This is the result of the estimate by the method of steepest descent for large $g \sim G$.

The second integral makes a contribution that is $\sqrt{g}$ times greater. Hence in the limit of large g, taking into account (8.2), we have

$$c_g^+ = \frac{\exp[\xi_0(1+\lambda)]}{\pi G^2} \frac{1-\lambda}{(1+\lambda)^2 \sqrt{\lambda}} \left(1 - \frac{1}{t}\right) \tag{8.7}$$

As $(t \to \infty)$ the final sediment spectrum is the same as that determined for the preceding model (see (6.10)).

9. The qualitative picture of the coagulation process in the models considered is as follows. At the initial stage, when the effect of the sink has not made itself felt, the asymptotic form of the spectrum of the active fraction for large g is defined by the gamma distributions /1, 2/

$$c_g^- \sim g^{-\gamma} \exp[-\alpha(t)\, g] \tag{9.1}$$

For the models in the order they were considered $\gamma = 0,\ ^3/_2,\ ^5/_2$ and $\alpha(t) = t^{-1}, e^{-4t}$, and $\ln t^{-1} - t\ (t \leqslant 1$ for the last model.

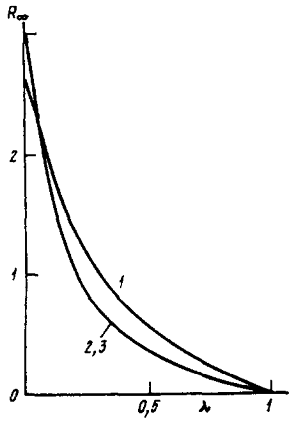

Fig.1

For the initial stage the condition $\alpha(t)G \ll 1$ of low concentration of particles of mass $g \sim G$ is characteristic. From this follows the estimate of the time of the initial period $t_1 \sim G$ and $\ln G, 1$, respectively.

In the transition period the spectrum of the active fraction becomes exponential: $c_g^- \sim g^{-\gamma}$. Then $\alpha(t)\, G \sim 1$ from which the esimate of the duration of the transition period is $t_2 \sim G, \ln G, G^{-1/2}$ (to obtain this result it is necessary to expand $\alpha(t)$ in series in the neighbourhood of $t = 1$). During this period intensive formulation of sediment begins.

Finally, the concluding stage is the formation of the final sediment spectrum. At this stage the effect of the sink is rather curious. The exponential mass spectrum of the active fraction is modified exponentially with respect to the growing factor $\exp(\xi_0 g/G)$ in the case of the last two models. In the model with constant kernel the spectrum of the active fraction increases exponentially and gradually concentrates near $g \sim G$.

The third model deserves a separate comment. It has already been mentioned that in this model a superparticle is formed, when $G = \infty$ and $t > 1$ /4/. However, the formal transition to $G \to \infty$ in the formulas of Sect.8 does not provide anything like that. This is not surprising, since the thermodynamic limit was considered (the particle concentration is finite and the total particle number in the volume of the system is infinite). Meanwhile, to establish the fact that a superparticle appears it is necessary to consider finite systems (see /4/), and superparticle formation can be followed by passing to the limit $G \to \infty$, but $0 < G/N < \infty$. Here N is the total number of particles in the coagulating system. In the above investigation the ratio $G/N = 0$ and, naturally, the passage to the limit necessary for detecting the superparticle, is impracticable.

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